JOURNAL OF APPROXIMATION THEORY 32, 170–188 (1981)

Improved Lower and Upper Bounds in Polynomial Chebyshev Approximation Based on a Pre-iteration Formula*

MANFRED HOLLENHORST

Hochschulrechenzentrum, Heinrich-Buff-Ring 44, D6300 Gießen, West Germany

Communicated by G. Meinardus

Received January 9, 1980

1. INTRODUCTION

The pre-iteration formulae of Meinardus [7] (see also [8, p. 121]) relate to certain functions f (f continuous on $I := [-1, 1]$) and to $n+2$ points $x_{k,f}$ which serve as an initial reference for the Remez algorithm in the Chebyshev approximation by polynomials of degree n. We shall choose in the following $n+2$ points

$$-1 = \xi_{0,f} < \xi_{1,f} < \cdots < \xi_{n+1,f} = 1$$

which are situated very close to the points $x_{k,f}$ obtained by the pre-iteration formulae, and under certain assumptions we determine an asymptotic expression for the values $L_{n,f}(f)$ of the linear functionals $L_{n,f}$ that satisfy

$$\begin{aligned} &L_{n,f}(g) = \sum_{k=0}^{n+1} \alpha_{k,f}\, g(\xi_{k,f}) \qquad \text{for } g \text{ continuous on } I, \\ &\|L_{n,f}\| = 1, \qquad\qquad (1) \\ &L_{n,f}(p) = 0 \qquad \text{for every polynomial } p \text{ of degree} \leqslant n. \end{aligned}$$

Under stronger hypotheses we can also show that $L_{n,f}(f)$ is greater in modulus than

$$L_n(f) := \frac{1}{n+1}\left[\sum_{k=1}^{n} (-1)^k f\left(\cos\frac{k\pi}{n+1}\right) + \frac{1}{2}\left((-1)^{n+1} f(-1) + f(1)\right)\right].$$

* Theorem 1 of this article is contained in the author's doctoral dissertation [5] written at the University of Erlangen under the direction of Prof. Dr. G. Meinardus.

0021-9045/81/070170-19$02.00/0

This means that in these cases $|L_{n,f}(f)|$ is a better lower bound for the minimum deviation

$$E_n(f) = \min\{\|f - p_n\|_\infty : p_n \text{ polynomial of degree } \leqslant n\}$$

with respect to the norm

$$\|g\|_\infty = \max_{t \in I} |g(t)|$$

(cf. Meinardus [8], p. 4).

Then we consider the polynomials $P_{n,f}$ and $p_{n,f}$ of minimal maximum deviation from f in the points $\xi_{k,f}$ $(k = 0, 1, ..., n+1)$ or $\xi_k := -\cos(k\pi/(n+1))$ $(k = 0, 1, ..., n+1)$, respectively, and under fairly restrictive assumptions we prove

$$\|f - P_{n,f}\|_\infty < \|f - p_{n,f}\|_\infty.$$

2. Pre-iteration Formulae

We consider a three times continuously differentiable function f on I with the development

$$f = \frac{c_0}{2} + \sum_{j=1}^{\infty} c_j T_j \tag{2}$$

with respect to Chebyshev polynomials of the first kind; if we truncate this development at $n + m + 1$ with an $m \in \{1, 2, ..., 2n+1\}$ and put

$$f^* = \frac{c_0}{2} + \sum_{j=1}^{n+m+1} c_j T_j$$

then we have (see, e.g., Hornecker [6])

$$e_n^* := f^* - p_{n,f^*} = c_{n+1} T_{n+1} + \sum_{j=1}^{m} c_{n+1+j}(T_{n+1+j} - T_{|n+1-j|}).$$

By a change of variables we obtain the equations

$$\begin{aligned}\tilde{e}_n(\phi) := e_n^*(\cos\phi) = c_{n+1} \cos[(n+1)\phi] \\ -2 \sum_{j=1}^{m} c_{n+1+j} \sin(j\phi) \sin[(n+1)\phi],\end{aligned}$$

$$\tilde{e}_n''\left(\frac{k\pi}{n+1}\right) = -(n+1)^2 c_{n+1}(-1)^k - 4(n+1)(-1)^k \sum_{j=1}^{m} j c_{n+1+j} \cos\frac{jk\pi}{n+1}.$$

Applying one step of the Newton iteration in order to determine the extrema of $\tilde{e}_n$ (starting with $\theta_k := k\pi/(n+1)$) and simplifying the denominator under the condition $c_{n+1} \neq 0$ we obtain

$$\theta_{k,f}^{(m)} := \theta_k - \frac{2\sum_{j=1}^{m} c_{n+1+j} \sin j\theta_k}{(n+1)\, c_{n+1}} \approx \theta_k - \frac{\tilde{e}_n'(\theta_k)}{\tilde{e}_n''(\theta_k)}$$
$$(k = 0, 1, \ldots, n+1).$$

Transforming back we get, in the simplest case of $m = 1$, the points

$$\begin{aligned} \xi_{k,f} &:= \cos\left(\theta_{n+1-k} - 2\,\frac{c_{n+2}}{c_{n+1}}\,\frac{\sin\theta_{n+1-k}}{n+1}\right) \\ &= \xi_k + 2\,\frac{c_{n+2}}{c_{n+1}}\,\frac{1-\xi_k^2}{n+1} \\ &\quad - 2\,\frac{1-\xi_k^2}{(n+1)^2}\left(\frac{c_{n+2}}{c_{n+1}}\right)^2 \cos(\theta_{n+1-k} + \eta_{n+1-k}), \end{aligned}$$

where η_{n+1-k} lies between 0 and

$$-2\,\frac{c_{n+2}}{c_{n+1}}\,\frac{\sin\theta_{n+1-k}}{n+1}.$$

If we carry out one step of the Newton iteration in order to determine the extrema of e_n^*, we obtain in the same way as above the reference points

$$x_{k,f}^{*(m)} = \xi_k + \frac{2(1-\xi_k^2)\sum_{j=1}^{m} c_{n+1+j}\, U_{j-1}(\xi_k)}{(n+1)\, c_{n+1} + \sum_{i=1}^{m} c_{n+1+i}(4iT_i(\xi_k) - 2\xi_k U_{i-1}(\xi_k))}$$
$$(k = 0, 1, \ldots, n+1), \tag{3}$$

where U_j denotes the Chebyshev polynomial of second kind and degree j. This is the general case of a pre-iteration formula of Meinardus [7; 8, p. 121]. In our theoretical investigations we consider only the approximate values

$$x_{k,f}^{(m)} = \xi_k + \frac{2(1-\xi_k^2)}{(n+1)\, c_{n+1}} \sum_{j=1}^{m} c_{n+1+j}\, U_{j-1}(\xi_k), \tag{3a}$$

which in the case $m = 1$ lie very close to the points $\xi_{k,f}$ as we have seen above.

3. Better Lower Bounds.

As usual we denote by I_j the Bessel function of order j (and of imaginary argument), i.e.,

$$I_j(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+j}}{k!\,(k+j)!}.$$

Then we have

THEOREM 1. *Let f with the development* (2) *and $c_{n+1} \neq 0$ be given, and let $\rho_n := 2(c_{n+2}/c_{n+1}) \neq 0$.*

(i) *If with some $\gamma \in \,]0, 1[$*

$$|c_{n+1+j}| \leqslant \gamma^j \, |c_{n+1}| \qquad \textit{for all natural numbers } j \tag{4}$$

holds, then

$$L_{n,f}(f) = \frac{c_{n+1} - \sum_{j=1}^{[2-\log n/\log \gamma]} c_{n+1+j}((-\rho_n)^j/j!) + c_{n+1}\gamma^2 O(1/n)}{I_0(\rho_n)} \times (1 + O(\gamma^n) + \rho_n^2 O(1/n)).$$

(ii) *If* (4) *holds and if $\gamma = |\rho_n|/2$, where $\gamma \leqslant 0.59 =: \gamma_0$, then for sufficiently large n we have*

$$|L_{n,f}(f)| > |L_n(f)|.$$

Remarks. (1) Statement (ii) means that in replacing the reference points

$$\xi_0, \xi_1, ..., \xi_{n+1}$$

by the reference points

$$\xi_{0,f}, \xi_{1,f}, ..., \xi_{n+1,f}$$

we achieve an "ascent" of the corresponding functionals from $L_n(f)$ to $L_{n,f}(f)$, just as if we had carried out one step of the Remez algorithm exactly. This result also holds for the points

$$\cos \theta_{0,f}^{(m)}, \cos \theta_{1,f}^{(m)}, ..., \cos \theta_{n+1,f}^{(m)}$$

with $m = 2$ or $m = 3$ instead of the points $\xi_{k,f}$ $(k = 0, 1, ..., n+1)$, if we again assume (4), $\gamma = |\rho_n|/2$ and (more restrictively than in the theorem) $\gamma \leqslant 0.23$ for $m = 2$, $\gamma \leqslant 0.26$ for $m = 3$ (see [5]).

(2) Condition (4) alone is satisfied for an infinite sequence of natural numbers n, if, e.g., f is holomorphic in an ellipse with foci -1 and $+1$ and a sum of half axes that is greater than γ^{-1}.

Proof of the theorem. In order to determine the "weights" $\alpha_{k,f}$ of $L_{n,f}$ we examine first

$$\tilde{\alpha}_{k,f} := \prod_{\substack{j=0\\ j\neq k}}^{n+1} (\xi_{k,f} - \xi_{j,f}).$$

Now by Taylor's formula

$$\xi_{k,f} = \xi_k + \frac{\rho_n}{n+1}(1-\xi_k^2) - \frac{\rho_n^2}{2(n+1)^2}\xi_k(1-\xi_k^2)$$
$$- \frac{\rho_n^3}{6(n+1)^3}(1-\xi_k^2)^2$$
$$+ \frac{\cos(\theta_{n+1-k} + \tilde{\eta}_{n+1-k})}{24(n+1)^4}(1-\xi_k^2)^2 \rho_n^4$$

holds with some $\tilde{\eta}_{n+1-k}$ between 0 and $-\rho_n(\sin\theta_{n+1-k}/(n+1))$. So we have

$$\xi_{k,f} - \xi_{j,f} = (\xi_k - \xi_j)\left(1 - \rho_n \frac{\xi_k + \xi_j}{n+1} + \frac{\rho_n^2}{(n+1)^2}\sigma_{j,k,n}\right),$$

where the $\sigma_{j,k,n}$ have bounds not depending on j, k or n. Therefore we have for $n \geqslant 5$ (see, e.g., Meinardus [8, p. 32])

$$\tilde{\alpha}_{k,f} = \prod_{\substack{j=0\\ j\neq k}}^{n+1} \left[(\xi_k - \xi_j)\left(1 - \frac{\rho_n}{n+1}(\xi_k + \xi_j)\right) \right.$$
$$\left. \times \left(1 - \frac{\rho_n^2 \sigma_{j,k,n}}{(n+1)^2\left(1 - \frac{\rho_n}{n+1}(\xi_k + \xi_j)\right)}\right)\right]$$
$$= \prod_{\substack{j=0\\ j\neq k}}^{n+1} (\xi_k - \xi_j) \frac{\prod_{i=0}^{n+1}\left(1 - \frac{\rho_n \xi_k}{n+1} - \frac{\rho_n}{n+1}\cos\frac{i\pi}{n+1}\right)}{\left(1 - \frac{2\rho_n \xi_k}{n+1}\right)} \left(1 + \rho_n^2 O\left(\frac{1}{n}\right)\right)$$

$$= \prod_{\substack{j=0\\ j\neq k}}^{n+1} (\xi_k - \xi_j)\, 2^{-n} \left(\frac{\rho_n}{n+1}\right)^{n+2} U_n \left(\frac{1 - \dfrac{\rho_n \xi_k}{n+1}}{\dfrac{\rho_n}{n+1}}\right) \left[\left(\frac{1 - \dfrac{\rho_n \xi_k}{n+1}}{\dfrac{\rho_n}{n+1}}\right)^2 - 1\right]$$

$$\times \frac{1 + \rho_n^2 O\left(\dfrac{1}{n}\right)}{1 - \dfrac{2\rho_n \xi_k}{n+1}}$$

$$= \prod_{\substack{j=0\\ j\neq k}}^{n+1} (\xi_k - \xi_j)\, 2^{-n-1}$$

$$\times \left\{\left[1 - \frac{\rho_n \xi_k}{n+1} + \sqrt{1 - \frac{2\rho_n \xi_k}{n+1} + \frac{\rho_n^2 \xi_k^2}{(n+1)^2} - \frac{\rho_n^2}{(n+1)^2}}\right]^{n+1}\right.$$

$$\left. - \left[1 - \frac{\rho_n \xi_k}{n+1} - \sqrt{1 - \frac{2\rho_n \xi_k}{n+1} + \frac{\rho_n^2 \xi_k^2}{(n+1)^2} - \frac{\rho_n^2}{(n+1)^2}}\right]^{n+1}\right\}$$

$$\times \left(1 + \frac{\dfrac{2\rho_n^2 \xi_k^2}{(n+1)^2} - \dfrac{\rho_n^2}{(n+1)^2}}{1 - \dfrac{2\rho_n \xi_k}{n+1}}\right)^{1/2} \frac{1 + \rho_n^2 O\left(\dfrac{1}{n}\right)}{\left(1 - \dfrac{2\rho_n \xi_k}{n+1}\right)^{1/2}}$$

$$= \prod_{\substack{j=0\\ j\neq k}}^{n+1} (\xi_k - \xi_j)\, 2^{-n-1}$$

$$\times \left\{2^{n+1} \left(1 - \frac{\rho_n \xi_k}{n+1} + \rho_n^2 O\left(\frac{1}{n^2}\right)\right)^{n+1} - \left[\rho_n^2 O\left(\frac{1}{n^2}\right)\right]^{n+1}\right\}$$

$$\times \left(1 - \frac{\rho_n \xi_k}{n+1} + \rho_n^2 O\left(\frac{1}{n^2}\right)\right)^{-1} \left(1 + \rho_n^2 O\left(\frac{1}{n}\right)\right)$$

$$= \prod_{\substack{j=0\\ j\neq k}}^{n+1} (\xi_k - \xi_j)\, e^{-\rho_n \xi_k} \left(1 + \rho_n^2 O\left(\frac{1}{n}\right)\right) \left(1 - \frac{\rho_n \xi_k}{n+1}\right)^{-1}$$

The last equality is a consequence of

$$\left(1 + \frac{\rho_n \xi_k}{n+1}\right)^{n+1} = e^{\rho_n \xi_k} e^{-(\rho_n \xi_k)^2/2(n+1) + (\rho_n \xi_k)^3/3(n+1)^2 - \cdots}.$$

Now we determine

$$\sum_{k=0}^{n+1} |\tilde{\alpha}_{k,f}|^{-1}$$
$$= \left[\sum_{k=1}^{n} \frac{e^{\rho_n \xi_k}}{n+1} \left(1 - \frac{\rho_n \xi_k}{n+1}\right) + \frac{e^{\rho_n}\left(1 - \frac{\rho_n}{n+1}\right) + e^{-\rho_n}\left(1 + \frac{\rho_n}{n+1}\right)}{2n+2} \right]$$
$$\times \left(1 + O\left(\frac{\rho_n^2}{n}\right)\right)$$
$$= \frac{1}{\pi} \int_0^{\pi} e^{\rho_n \cos\phi} \left(1 - \frac{\rho_n}{n+1} \cos\phi\right) d\phi \, (1 + O(\gamma^n)) \left(1 + \rho_n^2 O\left(\frac{1}{n}\right)\right)$$
$$= \left(I_0(\rho_n) + \frac{\rho_n}{n+1} I_1(-\rho_n)\right)\left(1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)\right)$$
$$= I_0(\rho_n) \left(1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)\right).$$

Then we have

$$\alpha_{k,f} = \frac{e^{-\rho_n \xi_k}(-1)^{n+1-k}}{(n+1)\, I_0(\rho_n)} \left(1 + \frac{\rho_n \xi_k}{n+1}\right)\left(1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)\right)$$
$$(k = 1, 2, \ldots, n),$$

$$\alpha_{0,f} = \frac{e^{\rho_n}(-1)^{n+1}}{(2n+2)\, I_0(\rho_n)} \left(1 - \frac{\rho_n}{n+1}\right)\left(1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)\right),$$

$$\alpha_{n+1,f} = \frac{e^{-\rho_n}\left(1 + \frac{\rho_n}{n+1}\right)}{(2n+2)\, I_0(\rho_n)} \left(1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)\right).$$

In order to determine the $e_n(\xi_{k,f})$ we introduce the notations

$$\delta_k := -\rho_n \frac{\sin\theta_k}{n+1} \qquad (k = 1, \ldots, n)$$

and

$$\kappa := \left[2 - \frac{\log n}{\log \gamma}\right],$$

where $[a]$ denotes the integer part of the real number a. For $j = 1, \ldots, \kappa$,

$$\begin{aligned}
&\cos[(n+1+j)(\theta_k+\delta_k)] - \cos[(n+1-j)(\theta_k+\delta_k)] \\
&\quad = -2(-1)^k \sin[(n+1)\,\delta_k] \sin[j(\theta_k+\delta_k)] \\
&\quad = -2(-1)^k \sin(-\rho_n \sin\theta_k) \\
&\qquad \times \left[\sin(j\theta_k)\left(1 - j^2\rho_n^2 O\left(\frac{1}{n^2}\right)\right) + \cos(j\theta_k) j\rho_n O\left(\frac{1}{n}\right)\right]
\end{aligned}$$

holds, and therefore

$$\begin{aligned}
e_n(\xi_{k,f}) &= c_{n+1}(-1)^{n+1-k}\cos[(n+1)\,\delta_{n+1-k}] \\
&\quad - 2\sum_{j=1}^{\kappa} c_{n+1+j}(-1)^{n+1-k}\sin(j\theta_{n+1-k})\sin(-\rho_n \sin\theta_{n+1-k}) \\
&\quad + c_{n+1}\rho_n \sum_{j=1}^{\kappa} j\gamma^j O\left(\frac{1}{n}\right) + c_{n+1}\gamma^2 O\left(\frac{1}{n}\right),
\end{aligned}$$

where the last term results from

$$\sum_{j=\kappa+1}^{\infty} |c_{n+1+j}| \leqslant \frac{\gamma^{\kappa+1}}{1-\gamma} = \gamma^2 O\left(\frac{1}{n}\right).$$

Combining these results we have

$$\begin{aligned}
L_{n,f}(f) &= \sum_{k=0}^{n+1} \alpha_{k,f}(f(\xi_{k,f}) - p_{n,f}(\xi_{k,f})) \\
&= \frac{1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)}{\pi I_0(\rho_n)}
\end{aligned}$$

$$\begin{aligned}
&\times \left[\int_0^{\pi} e^{-\rho_n \cos\phi} c_{n+1}\cos(-\rho_n\sin\phi)\left(1 + \rho_n \frac{\cos\phi}{n+1}\right) d\phi (1 + O(\gamma^n)) \right. \\
&\quad - 2\sum_{j=1}^{\kappa}\int_0^{\pi} e^{-\rho_n\cos\phi} c_{n+1+j}\sin(-\rho_n\sin\phi)\sin j\phi\, d\phi\,(1 + O(\gamma^n)) \\
&\quad \left. + c_{n+1}\gamma O\left(\frac{1}{n}\right)\left(\rho_n \frac{1}{(1-\gamma)^2} + \gamma\right)\right]
\end{aligned}$$

$$= \frac{c_{n+1}\left(1+\frac{\rho_n^2}{2n+2}\right) - \sum_{j=1}^{\kappa} c_{n+1+j}\frac{(-\rho_n)^j}{j!} + c_{n+1}\gamma^2 O\left(\frac{1}{n}\right)}{I_0(\rho_n)}$$

$$\times \left(1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)\right),$$

according to formulae 3.931 and 3.932 in Gradshteyn and Ryzhik [4].

Proof of ii. As a consequence of the above equation we have

$$|L_{n,f}(f)| \geqslant |c_{n+1}| \frac{1 + \frac{\rho_n^2}{2} - e^{\gamma|\rho_n|} + 1 + \gamma|\rho_n| - \gamma^2 O\left(\frac{1}{n}\right)}{I_0(\rho_n)}$$

$$\times \left(1 + O(\gamma^n) + \rho_n^2 O\left(\frac{1}{n}\right)\right).$$

If we choose $\gamma_0 = 0.59$ and assume $|\rho_n| = 2\gamma_0$, then

$$\frac{2 + 4\gamma_0^2 - e^{2\gamma_0^2}}{I_0(2\gamma_0)} > 1.00485$$

holds, i.e.,

$$1 > \frac{1}{\gamma_0^2}\left(I_0(2\gamma_0) + e^{2\gamma_0^2} - 2 - 3\gamma_0^2\right).$$

Now

$$\frac{1}{\gamma^2}\left(I_0(2\gamma) + e^{2\gamma^2} - 2 - 3\gamma^2\right)$$

$$= \frac{1}{\gamma^2}\left[\left(1 + \gamma^2 + \frac{\gamma^4}{4} + \frac{\gamma^6}{36} + \cdots\right) + \left(1 + 2\gamma^2 + \frac{4\gamma^4}{2} + \frac{8\gamma^6}{6} + \cdots\right) - 2 - 3\gamma^2\right]$$

is a monotonically increasing function of γ for $\gamma \geqslant 0$, and accordingly for $0 < \gamma < \gamma_0$ and $|\rho_n| = 2\gamma$

$$|L_{n,f}(f)| \geqslant |c_{n+1}| \frac{2 + 4\gamma^2 - e^{2\gamma^2}}{I_0(2\gamma)}\left(1 - \gamma^2 O\left(\frac{1}{n}\right)\right)$$

$$> |L_n(f)| = \left|\sum_{r=0}^{\infty} c_{(2r+1)(n+1)}\right|$$

holds, if only n is sufficiently large.

As we could see in the proof, one obtains numerous similar representations of $L_{n,f}(f)$ and corresponding estimates by slightly varying the assumptions of the theorem.

4. Reduction of the Error Norm

Next we want to show, for certain functions, that the error norm decreases in the case of discrete Chebyshev approximation in the points $\xi_{k,f}$ $(k = 0, 1, \ldots, n+1)$ (compared with approximation in the ξ_k $(k = 0, 1, \ldots, n+1)$).

Theorem 2. *If for a function f with the development* (2) *and for a sequence $(n_j)_{j \in \mathbb{N}}$ of positive integers*

$$c_{n_j+1} \neq 0, \qquad c_{n_j+2} \neq 0,$$
$$|c_{n_j+1+k}| \leqslant M \, |\gamma_{n_j}|^k \, |c_{n_j+1}| \qquad \text{for} \quad k = 2, 3, \ldots \tag{5}$$

holds with some $M > 0$, where $\gamma_{n_j} = c_{n_j+2}/c_{n_j+1}$ and

$$\lim_{j \to \infty} \gamma_{n_j} = 0,$$

then for n_j sufficiently large we have

$$\|f - P_{n_j,f}\|_\infty < \|f - p_{n_j,f}\|_\infty .$$

Remark. From the assumptions of the theorem it follows that f is an entire function.

Proof. For the sake of simplicity we put $n_j = n$ in the proof, i.e., we consider only those degrees n, for which (5) holds. First we examine where the extrema of $e_n := f - p_{n,f}$ (within $[-1, 1]$) can lie and what values $f - p_{n,f}$ can take there. To this end we denote the extremum of e_n neighbouring $\xi_{k,f}$ by x_k^*, and we put

$$\theta_{n+1-k}^* = \arccos x_k^*$$

and

$$\theta_k^{(1)} = \theta_k - \frac{\hat{e}_n'(\theta_k)}{\hat{e}_n''(\theta_k)},$$

where $\hat{e}_n = e_n \circ \cos$. The conditions for determining θ_k^* as a zero of $\hat{e}_n'$ by the Newton method, starting with θ_k, are fulfilled, and we can estimate

$$
\begin{aligned}
&|x^*_{n+1-k} - \xi_{n+1-k,f}| \\
&\quad \leqslant |\theta_k^* - \theta_{k,f}| \leqslant |\theta_k^* - \theta_k^{(1)}| + |\theta_k^{(1)} - \theta_k - \delta_k| \\
&\quad \leqslant 2\,\frac{|\hat{e}_n'(\theta_k)|^2}{|\hat{e}_n''(\theta_k)|^3} \max \left\{ |\hat{e}_n'''(\Xi)| : |\Xi - \theta_k^{(1)}| \leqslant \left| \frac{\hat{e}_n'(\theta_k)}{\hat{e}_n''(\theta_k)} \right| \right\} \\
&\qquad + \left| \frac{2\gamma_n \sin\theta_k}{n+1} \right. \\
&\qquad \left. - \frac{2\sum_{j=1}^{2n+1} c_{n+1+j}\sin(j\theta_k) + c_{n+1} O(|\gamma_n|^{2n+2})}{(n+1)\,c_{n+1} + 4\sum_{j=1}^{2n+1} c_{n+1+j}\, j\cos(j\theta_k) + c_{n+1} O(|\gamma_n|^{2n+2})} \right| .
\end{aligned}
$$

We still need the estimate

$$
\begin{aligned}
|\hat{e}_n'''(\Xi)| &= \Big| -(n+1)^3 c_{n+1} \sin[(n+1)\Xi] - \sum_{v=1}^{2n+1} c_{n+1+v} \\
&\qquad \times \{(n+1+v)^3 \sin[(n+1+v)\Xi] - (n+1-v)^3 \sin[(n+1-v)\Xi]\} \\
&\qquad + O(|\gamma_n|^{2n+2})\, c_{n+1} \Big| \\
&\leqslant |c_{n+1}|\,(n+1)^3\, [|\sin[(n+1)(\Xi - \Theta_k)]| + O(\gamma_n^2)] \\
&\qquad + |c_{n+1}|\,|\gamma_n|\,|(n+2)^3 \sin[(n+2)\Xi] - n^3 \sin(n\Xi)| \\
&\leqslant |c_{n+1}|\,(n+1)^3\,|\gamma_n|\, 6(1 + O(|\gamma_n|) + O(1/n)),
\end{aligned}
$$

which holds for

$$
|\Xi - \theta_k^{(1)}| \leqslant 2\left| \frac{\hat{e}_n'(\theta_k)}{\hat{e}_n''(\theta_k)} \right| \leqslant 2\,\frac{2|\gamma_n| + O(\gamma_n^2) + O(|\gamma_n|/n)}{n+1}.
$$

Then we get

$$
\begin{aligned}
&|\theta_k^* - \theta_{k,f}| \\
&\quad \leqslant \left\{ 2\,\frac{4\gamma_n^2}{(n+1)^2}\, \frac{|c_{n+1}|\, 6(n+1)^3\, |\gamma_n|\, (1 + O(|\gamma_n|) + O(1/n))}{(n+1)^2\, |c_{n+1}|} \right. \\
&\qquad \left. + \frac{2M\gamma_n^2}{n+1}(1 + O(|\gamma_n|)) \right\} |\sin\theta_k| \\
&\quad \leqslant \frac{2M\gamma_n^2}{n+1}(1 + O(|\gamma_n|))\,|\sin\theta_k|.
\end{aligned}
$$

We now give an asymptotic expression for $\hat{e}_n(\theta_{k,f}+\beta_k)$ with $\beta_k=(O(\gamma_n^2)/(n+1))\sin\theta_k$, e.g., for $\hat{e}_n(\theta_k^*)$ and $\hat{e}_n(\theta_{k,f})$:

$$\begin{aligned}
&\hat{e}_n(\theta_{k,f}+\beta_k)\\
&\quad= c_{n+1}(-1)^k\left[1-\frac{(n+1)^2}{2}(\delta_k+\beta_k)^2+O((n+1)^4(\delta_k+\beta_k)^4)\right]\\
&\qquad-2\sum_{j=1}^{2n+1}c_{n+1+j}\Big\{\sin(j\theta_k)(-1)^k(\delta_k+\beta_k)(n+1)\\
&\qquad+(-1)^k(\delta_k+\beta_k)^2 j(n+1)\cos(j\theta_k)\\
&\qquad-(-1)^k\frac{(\delta_k+\beta_k)^3}{12}\langle(n+1+j)^3\sin[j\theta_k+(n+1+j)\eta_{k,j}]\\
&\qquad+(n+1-j)^3\sin[j\theta_k-(n+1-j)\eta_{k,j}]\rangle\Big\}+O(\gamma_n^{2n+2})c_{n+1}\\
&\quad= c_{n+1}(-1)^k\left[1-2\left(\frac{c_{n+2}}{c_{n+1}}\right)^2\sin^2\theta_k-\frac{(n+1)^2}{2}(2\delta_k\beta_k+\beta_k^2)+O(\gamma_n^4)\right.\\
&\qquad+4\left(\frac{c_{n+2}}{c_{n+1}}\right)^2\sin^2\theta_k+\beta_k\left(-2\frac{c_{n+2}}{c_{n+1}}\frac{\sin\theta_k}{n+1}\right)(n+1)^2+O\left(\frac{|\gamma_n|^3}{n+1}\right)\\
&\qquad\left.+O(\gamma_n^4)+M\sum_{j=2}^{2n+1}j\,|\gamma_n|^{j+1}\sin^2\theta_k+O(\gamma_n^4)\right]\\
&\quad= c_{n+1}(-1)^k\Big[1+2\gamma_n^2\sin^2\theta_k(1+O(|\gamma_n|))\\
&\qquad+O(\gamma_n^4)+O\left(\frac{|\gamma_n|^3}{n+1}\right)\Big],
\end{aligned}\tag{6}$$

where the $\eta_{k,j}$ lie betweenθ_k and $\theta_{k,f}+\beta_k$. Hence

$$\|f-p_{n,f}\|_\infty\geqslant|c_{n+1}|\,(1+2\gamma_n^2-O(|\gamma_n|^3)-O(\gamma_n^2/n^2))$$

follows.

Now we derive an asymptotic expression for the function $P_{n,f}-p_{n,f}$, and to this end we define

$$l_k(x):=\prod_{\substack{j=0\\ j\neq k}}^{n+1}\frac{(x-\xi_j)}{(\xi_k-\xi_j)},$$

$$l_{k,f}(x):=\prod_{\substack{j=0\\ j\neq k}}^{n+1}\frac{(x-\xi_{j,f})}{(\xi_{k,f}-\xi_{j,f})}.$$

With these definitions we have (cf. Meinardus [8, p. 73])

$$\begin{aligned} P_{n,f} - p_{n,f} = P_{n,f-p_{n,f}} &= \sum_{k=0}^{n+1} \left[e_n(\xi_{k,f}) - (-1)^{n+1-k} L_{n,f}(f) \right] l_{k,f} \\ &= \sum_{k=0}^{n+1} c_{n+1}(-1)^{n+1-k} \\ &\quad \times \gamma_n^2 \left[2(1-\xi_k^2)(1+O(|\gamma_n|)) - 1 + O(|\gamma_n|) + O\left(\frac{1}{n}\right) \right] l_{k,f}; \end{aligned} \tag{7}$$

for according to Theorem 1 for sufficiently small γ (in our case for n sufficiently large) the following holds:

$$\begin{aligned} L_{n,f}(f) &= c_{n+1} \left(1 + 2\gamma_n^2 + O(\gamma_n^4) + O\left(\frac{\gamma_n^2}{n}\right)\right) \\ &\quad \times (1 - \gamma_n^2 + O(\gamma_n^4)) \left(1 + O(|\gamma_n|^n) + O\left(\frac{\gamma_n^2}{n}\right)\right) \\ &= c_{n+1}(1+\gamma_n^2)\left(1 + O\left(\frac{\gamma_n^2}{n}\right) + O(\gamma_n^4)\right) \end{aligned}$$

We next determine the polynomial of degree $\leqslant n+1$ which interpolates $T_{n+1}(x)(1-2x^2)$ in the points ξ_k, i.e., in the zeroes of $U_n(x)(1-x^2)$. First we have with $x = \cos\phi$

$$\begin{aligned} T_{n+1}(x) - xU_n(x) &= \frac{\sin\phi\cos[(n+1)\phi] - \cos\phi\sin[(n+1)\phi]}{\sin\phi} \\ &= -U_{n-1}(x) \end{aligned}$$

and therefore

$$T_{n+1}(x)(1-2x^2) = 2xU_n(x)(1-x^2) - 2U_{n-1}(x)(1-x^2) - T_{n+1}(x)$$

and furthermore

$$\begin{aligned} -2U_{n-1}(x)(1-x^2) - T_{n+1}(x) &= -2\sin n\phi\sin\phi - \cos n\phi\cos\phi \\ &\quad + \sin n\phi \sin\phi \\ &= -T_{n-1}(x). \end{aligned}$$

We now introduce the function

$$\zeta(x) = \cos\left(\arccos x - \frac{2\gamma_n}{n+1}\sqrt{1-x^2}\right)$$

for which $\xi_{k,f} = \zeta(\xi_k)$ holds and we estimate, as in the case of the $\alpha_{k,f}$,

$$\left|\frac{l_{k,f}(\zeta(x))}{l_k(x)}\right| = \prod_{\substack{j=0\\ j\neq k}}^{n+1} \frac{\left|1 - 2\gamma_n \dfrac{x+\xi_j}{n+1} + O\left(\dfrac{\gamma_n^2}{n^2}\right)\right|}{\left|1 - 2\gamma_n \dfrac{\xi_j+\xi_k}{n+1} + O\left(\dfrac{\gamma_n^2}{n^2}\right)\right|}$$

$$\leqslant e^{8|\gamma_n|}\left(1 + O\left(\frac{|\gamma_n|}{n}\right)\right).$$

Therefore we have, for $x \in [-1, 1]$,

$$P_{n,f}(\zeta(x)) - p_{n,f}(\zeta(x))$$

$$= \sum_{k=0}^{n+1} c_{n+1}\gamma_n^2(-1)^{n+1-k}(1-2\xi_k^2)\, l_k(x)\left(1 + O(|\gamma_n|) + O\left(\frac{1}{n}\right)\right)$$

$$= -c_{n+1}\gamma_n^2\left(1 + O(|\gamma_n|) + O\left(\frac{1}{n}\right)\right) T_{n-1}(x).$$

In order to estimate $f - P_{n,f}$ we choose a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of positive numbers with $\Gamma_n \to 0$, $1/(\Gamma_n \cdot n) \to 0$, and $\gamma_n/\Gamma_n \to 0$, and we partition the interval $[-1, 1]$ as follows.

(a) Let $x \leqslant -1/\sqrt{2} - \Gamma_n$ or $x \geqslant 1/\sqrt{2} + \Gamma_n$. Then by (6) and (7) we have the simple estimate

$$\begin{aligned}
&|f(x) - P_{n,f}(x)| \\
&\quad\leqslant \max\{|f(x) - p_{n,f}(x)| : x < -1/\sqrt{2} - \Gamma_n \text{ or } x > 1/\sqrt{2} + \Gamma_n\} \\
&\qquad + \|P_{n,f} - p_{n,f}\|_\infty \\
&\quad\leqslant |c_{n+1}|\,[1 + 2\gamma_n^2(1 - \tfrac{1}{2} - 2\Gamma_n/\sqrt{2} - \Gamma_n^2) + O(|\gamma_n^3|) \\
&\qquad + \gamma_n^2(1 + O(|\gamma_n|) + O(1/n))\,\|T_{n-1}\|_\infty] \\
&\quad\leqslant |c_{n+1}|\,[1 + 2\gamma_n^2 - 4\gamma_n^2\Gamma_n/\sqrt{2} + O(|\gamma_n^3|) + O(\gamma_n^2/n) - 2\gamma_n^2\Gamma_n^2] \\
&\quad< \|f - p_{n,f}\|_\infty,
\end{aligned} \tag{8}$$

where (8) holds for all $n = n_j$ sufficiently large because of (6).

(b) In the interval $[-1/\sqrt{2}-\Gamma_n,\ 1/\sqrt{2}+\Gamma_n]$ we consider first the subintervals in which no extremum of e_n can lie. We observe

$$\hat{e}_n(\theta_k)=(-1)^k\sum_{r=0}^{\infty}c_{(2r+1)(n+1)}=(-1)^k c_{n+1}(1+O(\gamma_n^{2n+2}))$$

and (analogous to (6))

$$\begin{aligned}\hat{e}_n(\theta_k+2\delta_k)\\&=c_{n+1}(-1)^k\left[1-\frac{(n+1)^2}{2}(2\delta_k)^2+O((n+1)^4\delta_k^4)\right]\\&\quad+c_{n+2}(-1)^k\Big[-2(n+1)\sin(\theta_k)\,2\delta_k\\&\quad-2(n+1)\cos(\theta_k)(2\delta_k)^2+\frac{(2\delta_k)^3}{6}O(n^3)\Big]+c_{n+1}O(|\gamma_n^3|)\\&=c_{n+1}(-1)^k\left[1+(n+1)^2\delta_k^2(-2+2)+O\left(\frac{\gamma_n^2}{n}\right)+O(|\gamma_n|^3)\right].\end{aligned}$$

Hence for x between $\cos\theta_{k+1}$ and $\cos(\theta_k+2\delta_k)$ with

$$k\in\{[(n+1)(\tfrac{1}{4}-\Gamma_n)],[(n+1)(\tfrac{1}{4}-\Gamma_n)]+1,\ldots,[(n+1)(\tfrac{3}{4}+\Gamma_n)]\}$$

it follows that

$$\begin{aligned}|f(x)-P_{n,f}(x)|&\leqslant|\hat{e}_n(\arccos x)|+\|P_{n,f}-p_{n,f}\|_\infty\\&\leqslant|c_{n+1}|\,[1+\gamma_n^2(1+O(|\gamma_n|)+O(1/n))]\\&<\|f-p_{n,f}\|_\infty\end{aligned}$$

if only $n=n_j$ is sufficiently large.

(c) In order to estimate $f-P_{n,f}$ near those extrema which lie inside $[-1/\sqrt{2}+\Gamma_n,\ 1/\sqrt{2}-\Gamma_n]$, we examine $\cos(n-1)\phi$ for $\phi=\theta_k+\beta_k$ with $|\beta_k|=O(|\gamma_n|/n)$ and

$$k\in\{[(n+1)(\tfrac{1}{4}+\Gamma_n)],[(n+1)(\tfrac{1}{4}+\Gamma_n)]+1,\ldots,[(n+1)(\tfrac{3}{4}-\Gamma_n)]\}:\quad(9)$$

$$\begin{aligned}(-1)^k\cos[(n-1)\phi]\\&=\cos(2\theta_k)\cos[(n-1)\beta_k]+\sin(2\theta_k)\sin[(n-1)\beta_k]\\&\leqslant-(2\pi\Gamma_n(2/\pi)-O(1/n))(1-O(\gamma_n^2))+O(|\gamma_n|)<-3\Gamma_n.\quad(10)\end{aligned}$$

Here (10) holds, if $n = n_j$ is sufficiently large. For x lying between $\cos\theta_k$ and $\cos(\theta_k + 2\delta_k)$ with k as in (9) we have

$$\begin{aligned}\operatorname{sgn}(c_{n+1}(-1)^k) &= \operatorname{sgn}(f(x) - p_{n,f}(x)) = \operatorname{sgn}(f(x) - P_{n,f}(x)) \\ &= \operatorname{sgn}\{-c_{n+1}\cos[(n-1)\,\zeta^{-1}(x)]\} \\ &= \operatorname{sgn}(P_{n,f}(x) - p_{n,f}(x))\end{aligned}$$

and hence

$$|f(x) - P_{n,f}(x)| < |f(x) - p_{n,f}(x)|$$

if $n = n_j$ is sufficiently large; for if x lies between $\cos\theta_k$ and $\cos(\theta_k + 2\delta_k)$ then $\zeta^{-1}(x)$ is situated between

$$\cos(\theta_k - \delta_k + O(\gamma_n^2/(n+1)^2)) \text{ and } \cos(\theta_k + \delta_k + O(\gamma_n^2/(n+1)^2)).$$

(d) If at last x lies between $\cos\theta_k$ and $\cos(\theta_k + 2\delta_k)$ with

$$k \in \{[(n+1)(\tfrac{1}{4} - 3\Gamma_n)], [(n+1)(\tfrac{1}{4} - 3\Gamma_n)] + 1, ..., [(n+1)(\tfrac{1}{4} + \Gamma_n)]\}$$

or

$$k \in \{[(n+1)(\tfrac{3}{4} - \Gamma_n)], [(n+1)(\tfrac{3}{4} - \Gamma_n)] + 1, ..., [(n+1)(\tfrac{3}{4} + 3\Gamma_n)]\}$$

then we have again for $\phi = \theta_k + \beta_k$ with $|\beta_k| = O(|\gamma_n|/n)$

$$(-1)^k \cos[(n-1)\phi] \leqslant 6\pi\Gamma_n + O(1/n) + O(\gamma_n)$$

and hence

$$\begin{aligned}&(-1)^k\,(f(x) - P_{n,f}(x)) \\ &\qquad \leqslant c_{n+1}(1 + \gamma_n^2(1 + O(\Gamma_n)) + \gamma_n^2(6\pi\Gamma_n + O(1/n) + O(\gamma_n)) \\ &\qquad < \|f - p_{n,f}\|_\infty,\end{aligned}$$

if $n = n_j$ is sufficiently large.

5. Related Results

(i) Under relatively weak hypotheses (see [5]) on the coefficients c_k in (2) the norm of the homogeneous mapping A_n that relates to a function f the polynomial of best approximation (degree n) in the points $x_{k,f}^{(m)}$ (and of similar mappings) is bounded by the quantity

$$(2/\pi)\log(n+1)(1 + o(1)).$$

(ii) A different nonlinear method in polynomial approximation studied by the author in [5] is based on a rational approximation in the complex plane as given by Akhiezer [1] and applied to polynomial approximation by Darlington [3], Binh Lam and Elliott [2], Talbot [9], and Gutknecht and Trefethen [10]. In this method the coefficients $c_{n+1}, c_{n+2},..., c_{n+m}$ of (2) are used to determine, by some kind of "throwback," a polynomial $Q_{n,m,f}$ of degree/ n. Then under the conditions

$$0 < c_{n+m} \leqslant \gamma c_{n+m-1} \leqslant \cdots \leqslant \gamma^{m-2} c_{n+2} \leqslant \gamma^{m-1} c_{n+1}$$

(with some γ, $0 < \gamma < 1$), $\lim_{n\to\infty} m(n) = \infty$, and $m(n) = o(n/\log n)$, the following results hold:

(a) If there is a $\delta < 1$ such that for all natural numbers k

$$|c_{n+k}| \leqslant \delta^{k-1} |c_{n+1}|$$

is valid then

$$\|f - Q_{n,m,f}\|_\infty = E_n(f)(1 + o(1)).$$

(b) If n is large enough and all c_k with $k > n$ are nonnegative then

$$\|f - Q_{n,m,f}\|_\infty < \left\| f - \frac{c_0}{2} - \sum_{k=1}^{n} c_k T_k \right\|_\infty.$$

6. Numerical Examples

The author has written a program to compute the polynomials $P^*_{n,f}$ of minimal maximum deviation from a function f in the points $x^{*(1)}_{k,f}$ ($k = 0, 1,..., n + 1$) as defined in (3). He computed these polynomials, which are a slight modification of the polynomials $P_{n,f}$ studied in this article, for a number of functions f and degrees n on the CD 3300 computer of the University of Erlangen; some results were already mentioned in [5].

We define, for $g \in \mathscr{C}(I)$, the functional

$$L^*_{n,f}(g) = \sum_{j=0}^{n+1} \frac{g(x^{*(1)}_{j,f})}{\prod_{k=0,k\neq j}^{n+1} (x^{*(1)}_{j,f} - x^{*(1)}_{k,f})} \times \left(\sum_{m=0}^{n+1} (-1)^{n+1-m} \prod_{\substack{i=0\\ i\neq m}}^{n+1} (x^{*(1)}_{i,f} - x^{*(1)}_{m,f})^{-1} \right)^{-1}$$

analogously to (1), so we can compare the quantities $|L_n(f)|$, $|L^*_{n,f}(f)|$, $E_n(f)$, $\|f - P^*_{n,f}\|_\infty$, and $\|f - p_{n,f}\|_\infty$ as is done in Table I for the functions e^t, $1/(t-2)$, and $(\arccos t)^2$ for several degrees. In addition to the asymptotic results of Theorems 1 and 2 one can see from the examples in Table I and from counterexamples (e.g., in [5]) that the condition

$$|c_{n+1}| > |c_{n+2}| > \cdots > |c_{n+m+1}| \tag{11}$$

TABLE I

Degree	$\lvert L_n(f)\rvert$	$\lvert L^*_{n,f}(f)\rvert$	$E_n(f)$	$\lVert f-P^*_{n,f}\rVert_\infty$	$\lVert f-p_{n,f}\rVert_\infty$
			Function $f(t)=e^t$		
1	2.7154032×10^{-1}	2.7880056×10^{-1}	2.7880159×10^{-1}	2.7880261×10^{-1}	2.8606285×10^{-1}
2	4.4336861×10^{-2}	4.5016943×10^{-2}	4.5017388×10^{-2}	4.5017847×10^{-2}	4.5468331×10^{-2}
3	5.4742404×10^{-3}	5.5283689×10^{-3}	$5.528369\ \times10^{-3}$	5.5283693×10^{-3}	5.5811151×10^{-3}
4	5.4292626×10^{-4}	5.4666751×10^{-4}	$5.466675\ \times10^{-4}$	5.4666762×10^{-4}	5.5008784×10^{-4}
5	4.4977313×10^{-5}	4.5205490×10^{-5}	$4.52055..\times10^{-5}$	4.5205810×10^{-5}	4.5429276×10^{-5}
6	3.1983973×10^{-6}	3.2108692×10^{-6}	$3.21090..\times10^{-6}$	3.2109500×10^{-6}	3.2229366×10^{-6}
7	1.9928211×10^{-7}	1.9979490×10^{-7}	$1.998....\times10^{-7}$	2.0047999×10^{-7}	2.0052078×10^{-7}
			Function $f(t)=1/(t-2)$		
2	2.2222222×10^{-2}	2.3932257×10^{-2}	2.3932257×10^{-2}	2.3932257×10^{-2}	2.5528196×10^{-2}
4	1.5948963×10^{-3}	1.7178220×10^{-3}	1.7182587×10^{-3}	1.7188785×10^{-3}	1.8384723×10^{-3}
6	1.1450818×10^{-4}	1.2330270×10^{-4}	1.2336543×10^{-4}	1.2343051×10^{-4}	1.3125171×10^{-4}
8	8.2213301×10^{-6}	8.8512053×10^{-6}	8.8572392×10^{-6}	8.8658615×10^{-6}	9.4877130×10^{-6}
			Function $f(t)=(\arccos t)^2$		
2	0.54831136	0.82056792	0.83760674	0.85820157	1.1640088
4	0.19739209	0.38149991	0.43412763	0.50178355	0.70730172
6	0.10071025	0.22186020	0.29158234	0.38035135	0.51152046
8	0.060923484	0.14519486	0.21927002	0.31198496	0.40124866

is essential for the fact that the points $x_{k,f}^{*(1)}$ yield a better approximation than the "Chebyshev nodes" ξ_k. On the other hand, given (11), the "regularity" properties of f (e.g., differentiability, holomorphy) determine how much $|L_{n,f}^{*}(f)|$ and $\|f-P_{n,f}\|_\infty$ deviate from $E_n(f)$. So in the case of functions that are neither even nor odd one should use as an initial reference for the Remez algorithm the points $x_{k,f}^{*(1)}$ if $|c_{n+2}| < |c_{n+1}|$ and the points ξ_k otherwise. In many examples (see Table I and also Meinardus [7]) the quantities $\|f-P_{n,f}^{*}\|_\infty$ and $E_n(f)$ agree so well that no step of the Remez iteration is needed. In the case of even or odd functions (which could be treated analogously in theory and computation) one should choose the points $x_{k,f}^{*(2)}$ if $|c_{n+3}| < |c_{n+1}|$ and the points ξ_k otherwise.

References

1. N. I. Akhiezer, Ob odnoj minimum-probleme teorii funkzij i o čisle kornej algebraičeskogo urawnenija, kotorie ležat wnutri ediničnogo kruga, *Izv. Akad. Nauk SSSR Otd. Mat. Estestv. Nauk* (1931), 1169–1189.
2. Binh Lam and D. Elliott, On a conjecture of C. W. Clenshaw, *SIAM J. Numer. Anal.* **9** (1972), 44–52.
3. S. J. Darlington, Analytical approximation to approximations in the Chebyshev sense, *Bell System Techn. J.* **49** (1970), 1–32.
4. I. S. Gradshteyn and I. M. Ryzhik, "Tables of Integrals, Sums, Series and Products," 4th ed., Academic Press, New York/London, 1965.
5. M. Hollenhorst, "Nichtlineare Verfahren bei der Polynomapproximation," Dissertation, Universität Erlangen–Nürnberg, 1976.
6. G. Hornecker, Méthodes pratiques pour la détermination approchée de la meilleure approximation polynômiale ou rationelle, *Chiffres* **4** (1960), 193–228.
7. G. Meinardus, Über Tschebyscheffsche Approximationen, *Arch. Rational Mech. Anal.* **9** (1962), 329–351.
8. G. Meinardus, "Approximation of Functions: Theory and Numerical Methods," Springer-Verlag, Berlin/Heidelberg/New York, 1967.
9. A. Talbot, The uniform approximation of polynomials by polynomials of lower degree, *J. Approx. Theory*, **17** (1976), 254–279.
10. M. H. Gutknecht and L. N. Trefethen, Real polynomial Chebyschev approximation by the Carathéodory–Féjèr method, *SIAM J. Numer. Anal.*, in press.